Kurmanbay Damir

«Soliton deformations of inversional minimal surfaces»

thesis for a degree of Doctor of Philosophy (Ph.D.) in specialty

6D060100 – Mathematics

ANNOTATION

The study's relevance. Soliton theory developed after the discoveries of Gardner, Green, Kruskal and Miura, the inverse scattering transformation for the Korteweg-de Fries (KdV) equation. They were led to this by the earlier discovery of solitons by Kruskal and Zabuska, who studied the Fermi-Pasta-Ulam problem for one-dimensional lattices. Thus began the modern development of soliton theories. But the first historical point of this discovery began 131 years ago.

In 1834 a marine engineer, John Scott Russell, was riding to the Union Canal near Edinburgh. In [Russell J. Scott, Fourteenth Meeting of the British Association for the Advancement of Science. Report of the Waves. - 1844 - London] he described as follows, what he saw at that moment:

"I noticed the motion of a boat being dragged by two horses in a thin channel, and when the boat suddenly stopped, -the water in the channel which was chasing the boat gathered in the head of the boat in a state of violent oscillation, and then it left the boat behind at great speeds, in the form of large unbound hills, rounded, flat and transparent, the water moving in a clear direction along the channel in its direction, without diminishing its shape or speed. On the top I caught up with it, which retained its original shape of about 9 meters in length, slid at a height of 30-45 cm at a speed of 14 km / h. Then its height gradually decreased, and after 2-3 km, I lost it in the channel. Thus, in 1834 I saw for the first time a unique and beautiful phenomenon and called it a wave of translation».

The modified Veselov-Novikov equation (mVN, which is two dimensional modified Korteweg-de Fries equation) was introduced in 1987 [Bogdanov L.V., Theor. Math. Phys.], and soliton mVN - deformations were introduced in [Konopelchenko B.G., Stud. Appl. Math., 1996], [Taimanov I.A., Amer. Math. Soc. Transl., Ser., 1997]. The algorithm for solving the modified Veselov-Novikov equation (mVN) was given in [Delong Yu, Q.P. Liu, Shikun Wang, J. of Physics, 2001] and a geometric interpretation of this method, called the Moutard transformation, was obtained in [Taimanov I.A., Math. Notes, 2015].

This transformation is given by the solution of the Dirac equation

$$\mathcal{D}\psi = 0$$

and three real constants, where $\mathcal{D} = \begin{pmatrix} U & \frac{\partial}{\partial z} \\ -\frac{\partial}{\partial \bar{z}} & V \end{pmatrix}$ - is the Dirac operator with potentials U, V. And any solution to the Dirac equation determines a surface in three-dimensional Euclidean space, given to the exact shifts, by the Weierstrass representation. On this surface a conformal parameter is given, and the potential U of the Dirac operator is the potential of the representation of this surface. By applying an inversion centered at the origin of coordinates to this surface, we obtain a new surface with the same conformal parameter and a new potential. Thus, in the dissertation work we invert only the minimal surface and, as a result, we obtain the soliton deformation of the inverted minimal surface. And the soliton deformation is determined by the potential representation of this surface, which is constructed by the Moutard transformation.

In 2014, blowing-up solutions of the mVN equation were obtained by rigid translation of the original Enneper surface in [Taimanov I.A., Math. Notes, 2015]. Further results were obtained for the second-order Enneper surface in [Kurmanbaev D.M., Bulletin of KazNU, ser. mech., math., inf., - 2015 - 84 - №1 - 77-86], [Geometry of manifolds and its aaplications: the fourth scientific conference with international participation, Ulan-Ude, Lake Baikal – Lake Schuchye, June 26 – June 30, 2016]. And in [Kurmanbaev D.M., Abstracts of papers devoted to the 80th anniversary of academician N.K. Bliev, Almaty, Kazakhstan, 15.10.2015-16.10.2015] were obtained blowing-up solutions of the mVN equation for the Ennepper surface of the third order, and in [News of the National Academy of sciences of the Republic of Kazakhstan, Series Physics and Mathematics, vol. 3, № 307 (2016), 163-167] regular solutions were found. But the soliton deformation for the high order Enneper surface for an arbitrary region is still unknown. Nevertheless, a soliton deformation has been found on some segment in the works [Kurmanbayev D., International Journal of Mathematics and Mathematical Sciences. - 2020. - ID 9740638. - Volume 2020. 1-7], [Kurmanbaev D.M., International scientific conference "Actual problems of mathematics and informatics", dedicated to the 80th anniversary of K.K. Kasymov, Academician of the National Academy of Sciences of Republic of Kazakhstan. - Almaty, Kazakhstan, 21 -23 December 2015. - P. 81-83] by a smooth parallel translation of the original high order Enneper surface. Further, in the dissertation work the soliton deformation for the inverted catenoid [Kurmanbayev D., Yesmakhanova K., Soliton deformation of inverted catenoid. // News of the National Academy of sciences of the Republic of Kazakhstan, vol. 2, №336. - 2021. - P. 24-32. - DOI:10.32014/2021.2518-1726.17.] and helicoid are found by smooth and rigid translation (respectively) along the second coordinate axis.

Moutard transformations have been successfully studied in the study of integrable systems in mathematical physics and differential geometry, in spectral theory and in complex analysis. Therefore, at present, the problems arising in the theory of solitons, especially with a geometrical interpretation, are relevant.

Purpose of the thesis: to find solutions of the modified Veselov-Novikov equation and to construct soliton deformations for inversional minimal surfaces in three-dimensional space.

In accordance with the goal, the following **research objectives** were set:

- applying the Moutard transformation for the modified Veselov-Novikov equation, find exact solutions of the mVN equation;

- construct regular and blowing-up solutions of the mVN equations by deformations such as smooth translation-fixing and rigid translation-fixing along some coordinate axis (straight line) of the initial minimal surfaces (catenoid, helicoid and high-order Enneper surfaces).
- compare the pictures of regular and blowing-up potentials of the representation of inverted minimal surfaces;
- analyze solutions of the mVN equations to relate the soliton deformation for inverted minimal surfaces by known methods of differential geometry and soliton theory.
- illustrate the application of soliton deformation for inversional minimal surfaces.

The object of study is initial minimal surfaces.

The subject of study is differential geometry, soliton theory.

Research Methods. The thesis uses the Moutard transformation to solve the modified Veselov-Novikov equation. Since the solutions of the mVN equation have a large form, these solutions are easily verified using the Maple software. And also these solutions make it possible to construct soliton mVN deformations. We propose a new method from differential geometry theory to study soliton deformation for inverted minimal surfaces.

The scientific novelty of the dissertation work:
- found solutions of the mVN equation on examples of high-order Enneper surface;
- found solutions of the modified Korteweg-de-Fries equation (mKdF) on the example of a catenoid;
- found solutions of the mVN equation on the example of a helicoid;
- soliton deformations for inversional minimal surfaces are constructed.

Theoretical and practical significance of the research. The present work contains theoretical as well as practical problems. The obtained results can be used in modern geometrical structures and in the theory of solitons. Also, the results of the work may have a practical character in solving various problems in theoretical physics.

The scientific statements made for the defense:
- construction of soliton deformation for high order inverted Enneper surfaces;
- finding a solution to the modified Korteweg-de-Fries equation (mKdF) by constructing a soliton deformation for an inverted catenoid;
- applying the soliton deformation to high order inverted Enneper surfaces, comparing the picture of regular and blowing-up potentials representation of these surfaces.

Approbation of dissertation work.

The main results of the thesis were presented and discussed at the following international scientific conferences:
- International scientific conference "Theory of functions, informatics, differential equations and their applications". Theses of reports dedicated to the 80th anniversary of academician of the National Academy of Sciences of the Republic of Kazakhstan N.K. Bliev, October 15-16, 2015, Almaty, Kazakhstan;
- International scientific conference "Actual problems of mathematics and informatics", dedicated to the 80th anniversary of academician of NAS of the

Republic of Kazakhstan K.K. Kasymov, 21-23 December 2015, Almaty, Kazakhstan;
- Third International Conference on Analysis and Applied Mathematics, *ICAAM 2016,* September 7-10, Almaty, Kazakhstan;
- International scientific conference "Geometry of manifolds", Russia, Republic of Buryatia, Ulan-Ude, June 27-30, 2016;
- VI scientific conference with international participation "Geometry of manifolds and its applications" Ulan-Ude - Lake Baikal, July 8-10, 2020;
also positively evaluated at the city scientific seminar of the Faculty of Mechanics and Mathematics of the Al-Farabi Kazakh National University "Differential operators and their applications", 2021.

Publications. On the topic of the dissertation work, the author published 9 publications, including 3 articles - in the journals recommended by the Committee for Control in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan, 1 article - in foreign scientific journals with a non-zero impact factor, included in the Scopus database, 1 article - in the refereed materials of VI International Scientific Conference, indexed by the RSCI base, 4 thesis - on the materials of international conferences.

The structure and scope of the thesis. The thesis consists a title page, content, list of symbols and abbreviations, introduction, three sections and their subsections, conclusion, list of references - 51. The total volume of the dissertation is 84 pages.

The main content of the dissertation. Each section of the thesis consists of subsections.

The **introduction** analyzes the current state of the research problem and gives a review of the literature, justifies the relevance of the topic, sets goals, defined research objectives, subject and object of study. The results of the research, their scientific novelty, theoretical and practical significance are presented.

The **first** section of the dissertation introduces the basic concepts of differential geometry in accordance with the current topics and uses the results of known work.

In the **second** section of the thesis, the well-known Moutard transformation (Darboux transformation), for solving the modified Veselov-Novikov equation, is given. And to represent the soliton deformation for inverted minimal surfaces, nontrivial solutions of the mVN equations are found. Namely, the soliton deformation for high order inverted Enneper surfaces is found, a problem which is posed in [Taimanov I.A., Blowing up solutions of the modified Novikov-Veselov equation and minimal surfaces, Math. Notes, 2015], and also soliton deformations for the inverted catenoid and inverted helicoid are investigated.

In the **third** section of the thesis solutions of the modified Veselov-Novikov equation are verified by the Maple software. Soliton deformation of pictures for high-order inverted Enneper surfaces are obtained, comparing the dynamics of motion of regular and blowing-up potential representations of these surfaces. Further illustrates the applications of soliton deformations to high order inverted Enneper surfaces.

The **conclusion** summarizes the main results and conclusions of the thesis research.